

Stochastic order characterization of uniform integrability and tightness

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Abstract

We show that a family of random variables is uniformly integrable if and only if it is stochastically bounded in the increasing convex order by an integrable random variable. This result is complemented by proving analogous statements for the strong stochastic order and for power-integrable dominating random variables. Especially, we show that whenever a family of random variables is stochastically bounded by a p -integrable random variable for some $p > 1$, there is no distinction between the strong order and the increasing convex order. These results also yield new characterizations of relative compactness in Wasserstein and Prohorov metrics.

Keywords: stochastic order, uniformly integrable, tight, stochastically bounded, bounded in probability, strong order, increasing convex order, integrated survival function, Hardy–Littlewood maximal random variable

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1 Introduction

Let $\{X_n\}$ be a sequence of random variables such that $X_n \rightarrow X$ almost surely. Lebesgue’s classical dominated convergence theorem implies that

$$E|X_n - X| \rightarrow 0 \tag{1}$$

if there exists an integrable random variable Y such that

$$|X_n| \leq Y \quad \text{almost surely for all } n.$$

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A probabilistic version (e.g. [17, Thm. 9.1]) of the above condition is to require that

$$|X_n| \leq_{\text{st}} Y \quad \text{for all } n, \quad (2)$$

where \leq_{st} denotes the strong (a.k.a. usual) stochastic order [11, 14, 16]. It is well known that in general (2) is not necessary for (1). We will show that a sharp characterization can be obtained when the strong stochastic order in (2) is replaced by a weaker one. Namely, we will show that (1) holds if and only if there exists an integrable random variable Y such that

$$|X_n| \leq_{\text{icx}} Y \quad \text{for all } n, \quad (3)$$

where \leq_{icx} denotes the increasing convex stochastic order [14, 16].

More generally, our first main result, Theorem 1, shows that a family of random variables $\{X_n\}$ is uniformly integrable if and only if (3) holds. For almost surely convergent random sequences, this result yields the equivalence of (1) and (3) (e.g. [8, Prop 4.12]). From the analysis point of view, the characterization of uniform integrability in terms of the increasing convex order can be seen as a new way to represent domination in Lebesgue's dominated convergence theorem in the natural manner. What makes this result important in probability is that convex and increasing convex orders are intimately connected with the existence of martingales and submartingales with given marginals (e.g. Kellerer [9]; Hirsch and Yor [7] and references therein).

The second main result, Theorem 2, shows that when studying whether a family of random variables is stochastically bounded by a random variable in L^p for some $p > 1$, there is no need to distinguish between the strong order and the increasing convex order. This somewhat surprising result, which is a consequence of a Hardy–Littlewood maximal inequality, may open new ways to establishing strong stochastic bounds using tools of convex analysis.

The main results are complemented by Proposition 1, which states that $\{X_n\}$ is tight if and only if (2) holds for some (almost surely finite) random variable Y . This simple result is probably well known, because ‘bounded in probability’ and ‘stochastically bounded’ are commonly used as synonyms for ‘tight’. We have formulated it here explicitly in order to complete the big picture on various stochastic boundedness relationships in Theorem 3. The implication diagram in Figure 1 summarizes the findings and illustrates how these concepts are related to relative compactness with respect to Wasserstein and Prohorov metrics. Figure 1 provides a new unified view to earlier studies on lattice properties of increasing convex and strong stochastic orders (Kertz and Rösler [10]; Müller and Scarsini [13]).

The rest of this paper is organized as follows. Section 2 introduces definitions and notation. Section 3 discusses uniform integrability and tightness of positive random variables, and Section 4 extends the analysis to power-integrable random variables. Section 5 summarizes the main results in a diagram (Figure 1) and presents counterexamples confirming the sharpness of the implications in the diagram.

2 Definitions and notation

2.1 Uniform integrability and tightness

In general, we shall assume that $\{X_\alpha\}$ is a family of random variables with values in the d -dimensional Euclidean space \mathbb{R}^d , indexed by a parameter α taking values in an arbitrary set. The random variables do not need to be defined on a common probability space; instead, we assume that each X_α is a random variable defined on some probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$. To keep the notation light, we replace P_α by P , and denote the expectation with respect to P_α by E . In addition, we denote by $\mu_\alpha = P \circ X_\alpha^{-1}$ the distribution of X_α defined on the Borel sets of \mathbb{R}^d . When $d = 1$, the distribution function of X_α is defined by $F_\alpha(t) = P(X_\alpha \leq t)$, and the quantile function by $F_\alpha^{-1}(u) = \inf\{t : F_\alpha(t) \geq u\}$ for $u \in (0, 1)$.

A family of random variables $\{X_\alpha\}$ is called:

- *uniformly integrable* if $\sup_\alpha E|X_\alpha|1(|X_\alpha| > t) \rightarrow 0$ as $t \rightarrow \infty$,
- *uniformly p -integrable* if $\sup_\alpha E|X_\alpha|^p 1(|X_\alpha| > t) \rightarrow 0$ as $t \rightarrow \infty$,
- *tight* if $\sup_\alpha P(|X_\alpha| > t) \rightarrow 0$ as $t \rightarrow \infty$, and
- *bounded in L^p* if $\sup_\alpha E|X_\alpha|^p < \infty$.

Remark 1. All results in this article remain valid also in a slightly more general setting where \mathbb{R}^d is replaced by a complete separable metric space S satisfying the Heine–Borel property: a subset of S is compact if and only if it is bounded and closed. In this case we replace the norm of x by $|x| = \rho(x, x_0)$, where ρ is the metric and x_0 is an arbitrary reference point in S . For concreteness, we prefer to formulate the results for \mathbb{R}^d -valued random variables.

2.2 Stochastic orders

For real-valued random variables X_1 and X_2 , we denote $X_1 \leq_{\text{st}} X_2$ and say that X_1 is less than X_2 in the *strong stochastic order*, or X_1 is *st-bounded* by X_2 , if $E\phi(X_1) \leq E\phi(X_2)$ for all increasing¹ measurable functions $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$. It is well known [11, 14, 16] that the following are equivalent for any real-valued random variables X_1 and X_2 with distribution functions F_1 and F_2 :

- (i) $X_1 \leq_{\text{st}} X_2$.
- (ii) $1 - F_1(t) \leq 1 - F_2(t)$ for all $t \in \mathbb{R}$.
- (iii) $F_1^{-1}(u) \leq F_2^{-1}(u)$ for all $u \in (0, 1)$.

¹The symbol \mathbb{R}_+ refers to positive real numbers including zero. In general, we use the terms ‘positive’, ‘increasing’, and ‘less than’ as synonyms for ‘nonnegative’, ‘nondecreasing’, and ‘less or equal than’.

- (iv) There exist random variables \hat{X}_1 and \hat{X}_2 defined on a common probability space, distributed according to X_1 and X_2 , such that $\hat{X}_1 \leq \hat{X}_2$ almost surely.

For positive random variables X_1 and X_2 , we denote $X_1 \leq_{\text{icx}} X_2$ and say that X_1 is less than X_2 in the *increasing convex order*, or X_1 is *icx-bounded* by X_2 , if $E\phi(X) \leq E\phi(Y)$ for all increasing convex functions $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. A fundamental characterization (e.g. [14, Thm. 1.5.7]) of increasing convex stochastic orders states that

$$X_1 \leq_{\text{icx}} X_2 \quad \text{if and only if} \quad H_1(t) \leq H_2(t) \text{ for all } t \geq 0,$$

where

$$H_i(t) = E(X_i - t)_+ = \int_t^\infty (1 - F_i(t)) dt$$

denotes the *integrated survival function* of X_i , and $x_+ = \max(x, 0)$. Basic technical facts about integral survival functions are given in Appendix B.

Remark 2. Note that if $EX_2 = \infty$, then any positive random variable is icx-bounded by X_2 . Therefore to say that a random variable X_1 is icx-bounded by X_2 truly makes sense only when X_2 is integrable, in which case also X_1 is integrable.

Remark 3. The notion of increasing convex order can also be extended to random variables on the full real line having an integrable right tail. Because the main results of this article concern norms of random variables, for simplicity we prefer to restrict our terminology to positive random variables, using ‘integrable’ in place of ‘having an integrable right tail’ etc.

3 Uniform integrability and tightness of positive random variables

The following theorem, the first main result of this article, provides a sharp characterization of uniform integrability as stochastic boundedness in the increasing convex order. We say that a family of positive random variables $\{X_\alpha\}$ is *icx-bounded* by a random variable Y if $X_\alpha \leq_{\text{icx}} Y$ for all α .

Theorem 1. *The following are equivalent for any family of positive random variables $\{X_\alpha\}$ with distribution functions F_α :*

- (i) $\{X_\alpha\}$ is icx-bounded by an integrable random variable.
- (ii) $\{X_\alpha\}$ is uniformly integrable.
- (iii) $\lim_{t \rightarrow \infty} \sup_\alpha \int_t^\infty (1 - F_\alpha(u)) du = 0$.

Proof. (i) \implies (ii). Assume that $X_\alpha \leq_{\text{icx}} X$ for all α , where X has a finite mean. Then $E(X_\alpha - t)_+ \leq E(X - t)_+$ for all $t \geq 0$. Note that

$$x1(x > t) \leq 2(x - t/2)_+$$

for all x and t . Therefore,

$$EX_\alpha 1(X_\alpha > t) \leq 2E(X_\alpha - t/2)_+,$$

which implies that

$$\sup_\alpha EX_\alpha 1(X_\alpha > t) \leq 2E(X - t/2)_+.$$

Because X has a finite mean, we see by using dominated convergence that the right side above tends to zero as t grows, and $\{X_\alpha\}$ is hence uniformly integrable.

(ii) \implies (iii). Because

$$\int_t^\infty (1 - F_\alpha(u)) du = E(X_\alpha - t)_+ \leq EX_\alpha 1(X_\alpha > t)$$

for all $t \geq 0$ and all α , the uniform integrability of $\{X_\alpha\}$ implies (iii).

(iii) \implies (i). Define $H(t) = \sup_\alpha H_\alpha(t)$, where $H_\alpha(t) = E(X_\alpha - t)_+$ is the integrated survival function of X_α . Note that H_α is convex and $H_\alpha(t) + t \geq H_\alpha(0)$ for all $t \geq 0$ by Lemma 3 in Appendix B. As a consequence, the same properties are valid for H . Further, $H(t) \rightarrow 0$ as $t \rightarrow \infty$ by (iii). Therefore, by Lemma 3 we conclude that H is the integrated survival function of an integrable positive random variable X . By the definition of H , it follows that for all α ,

$$E(X_\alpha - t)_+ = H_\alpha(t) \leq H(t) = E(X - t)_+ \quad \text{for all } t \geq 0,$$

which is equivalent to $X_\alpha \leq_{\text{icx}} X$ [14, Thm. 1.5.7]. \square

Remark 4. The role of increasing convex functions in characterizing uniform integrability is also visible in the de la Vallée-Poussin theorem, which states that a family of random variables $\{X_\alpha\}$ in \mathbb{R}^d is uniformly integrable if and only if

$$\sup_\alpha E\phi(|X_\alpha|) < \infty$$

for some increasing convex function ϕ such that $\phi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$ [4, Prop. A.2.2].

The following simple result characterizes tightness as stochastic boundedness in the strong order. We say that a family of random variables $\{X_\alpha\}$ is *st-bounded* by a random variable Y if $X_\alpha \leq_{\text{st}} Y$ for all α . A positive random variable X (which by convention is allowed to take on the value ∞) is called *finite* if $P(X < \infty) = 1$.

Proposition 1. *The following are equivalent for any family of positive random variables $\{X_\alpha\}$ with distribution functions F_α :*

(i) $\{X_\alpha\}$ is st-bounded by a finite random variable.

(ii) $\{X_\alpha\}$ is tight.

(iii) $\lim_{t \rightarrow \infty} \sup_\alpha (1 - F_\alpha(t)) = 0$.

Proof. (i) \implies (ii). Assume that $X_\alpha \leq_{\text{st}} X$ for all α , where X is a finite random variable. Then

$$\sup_\alpha P(X_\alpha > t) \leq P(X > t).$$

Because the right side above tends to zero as $t \rightarrow \infty$, tightness follows.

(ii) \implies (iii). Clear.

(iii) \implies (i). Let $F(t) = \inf_\alpha F_\alpha(t)$. Then Lemma 2 in Appendix A shows that F is a distribution function of a finite positive random variable X_* . Because $1 - F_\alpha(t) \leq 1 - F(t)$ for all t and all α , it follows that $X_\alpha \leq_{\text{st}} X_*$ for all α . \square

Remark 5. An alternative equivalent characterization of tightness, analogous to Remark 4, is the following: A family of random variables $\{X_\alpha\}$ in \mathbb{R}^d is tight if and only if

$$\sup_\alpha E\phi(|X_\alpha|) < \infty$$

for some positive function ϕ such that $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$ [12, Lem. D.5.3].

4 Stochastic boundedness of power-integrable random variables

4.1 Boundedness in strong vs. increasing convex order

If a family of positive random variables is st-bounded by a p -integrable random variable for some $p > 0$, then the same is obviously true when ‘st-bounded’ is replaced by ‘icx-bounded’. The following result shows that, rather surprisingly, these two properties are equivalent when $p > 1$.

Theorem 2. *For any $p > 1$, a family of positive random variables $\{X_\alpha\}$ is icx-bounded by a p -integrable random variable if and only if $\{X_\alpha\}$ is st-bounded by a p -integrable random variable.*

To prove this result, we will apply Hardy–Littlewood maximal functions. Let X be an integrable random variable with distribution function F , and denote the quantile function of X by $F^{-1}(u) = \inf\{t : F(t) \geq u\}$ for $u \in (0, 1)$. Let U be a uniformly distributed random variable in $(0, 1)$ and define

$$X^* = MF^{-1}(U),$$

where

$$MF^{-1}(u) = \sup_{u < w < 1} (w - u)^{-1} \int_u^w F^{-1}(v) dv = (1 - u)^{-1} \int_u^1 F^{-1}(v) dv$$

is the Hardy–Littlewood maximal function of F^{-1} [6]. The function MF^{-1} is well-defined when X is integrable [3, 10]. We call X^* the *Hardy–Littlewood maximal random variable* associated with X . A remarkable result of Kertz and Rösler [10, Lem. 4.1] states that for any integrable random variables X and Y ,

$$X \leq_{\text{icx}} Y \quad \text{if and only if} \quad X^* \leq_{\text{st}} Y^*. \quad (4)$$

Lemma 1. $X \leq_{\text{st}} X^*$ for any integrable random variable X .

Proof. Observe that the quantile function of X^* equals MF^{-1} . Further, the monotonicity of F^{-1} shows that $F^{-1}(u) \leq MF^{-1}(u)$ for all $u \in (0, 1)$. This pointwise ordering of the quantile functions is a well-known necessary and sufficient condition for the strong ordering of the corresponding random variables (e.g. [14, Thm. 1.2.4]). \square

Proof of Theorem 2. Assume that $X_\alpha \leq_{\text{icx}} Y$ for all α , where $Y \in L^p$. Then $X_\alpha^* \leq_{\text{st}} Y^*$ for all α due to (4), so by applying Lemma 1 we see that the family $\{X_\alpha\}$ is st-bounded by Y^* . It remains to verify that Y^* is in L^p . Note that the quantile function of Y^* equals MF_Y^{-1} , where F_Y^{-1} is the quantile function of Y . A classic inequality of Hardy and Littlewood [6] now implies that

$$E(Y^*)^p = \int_0^1 (MF_Y^{-1}(u))^p du \leq c \int_0^1 (F_Y^{-1}(u))^p du = cEY^p,$$

where $c = \left(\frac{p}{p-1}\right)^p$. \square

4.2 Power-integrable strong bounds

The following result shows how to reduce the study of strong boundedness by a p -integrable random variable to strong boundedness by an integrable random variable. A corresponding statement is not true for the increasing convex ordering, see Theorem 3 in Section 5.

Proposition 2. For any $p > 0$ and any family of positive random variables $\{X_\alpha\}$ with distribution functions F_α , the following are equivalent:

- (i) $\{X_\alpha\}$ is st-bounded by a p -integrable random variable.
- (ii) $\{X_\alpha^p\}$ is st-bounded by an integrable random variable.
- (iii) $\int_0^\infty \sup_\alpha (1 - F_\alpha(t^{1/p})) dt < \infty$.

Proof. (i) \implies (ii). Assume that $X_\alpha \leq_{\text{st}} Y$ for all α , for some p -integrable random variable Y . Because the random variables X_α are positive, it follows that $Y \geq 0$ almost surely. Because $x \mapsto x^p$ is increasing on \mathbb{R}_+ , it follows that $X_\alpha^p \leq_{\text{st}} Y^p$ for all α .

(ii) \implies (i). The same argument as above works also in this direction, because $x \mapsto x^{1/p}$ is increasing.

(i) \implies (iii). Assume that $X_\alpha \leq_{\text{st}} Y$ for all α , for some p -integrable random variable $Y \geq 0$. Define $F(t) = \inf_\alpha F_\alpha(t)$. Proposition 1 implies that $\lim_{t \rightarrow \infty} F(t) = 1$, so by Lemma 2 in Appendix A, we see that F is a distribution function of a finite positive random variable Z . Because $Z \leq_{\text{st}} X_\alpha$ for all α , it follows that $Z \leq_{\text{st}} Y$. The monotonicity of $x \mapsto x^p$ further implies that $EZ^p \leq EY^p < \infty$. Therefore,

$$\int_0^\infty \sup_\alpha (1 - F_\alpha(t^{1/p})) dt = \int_0^\infty (1 - F(t^{1/p})) dt = EZ^p < \infty.$$

(iii) \implies (i). Define $F(t) = \inf_\alpha F_\alpha(t)$, and observe that $\lim_{t \rightarrow \infty} F(t) = 1$ by (iii). Again, by Lemma 2, we see that F is the distribution function of some finite positive random variable Z , and $X_\alpha \leq_{\text{st}} Z$ for all α . Further, the formula

$$EZ^p = \int_0^\infty (1 - F(t^{1/p})) dt$$

shows that Z is p -integrable. \square

We conclude this section by a simple corollary of the above result.

Proposition 3. *Let $0 < p < q$. If a family of positive random variables $\{X_\alpha\}$ is bounded in L^q , then it is st-bounded by a p -integrable random variable.*

Proof. Markov's inequality implies that

$$1 - F_\alpha(t^{1/p}) = P(|X_\alpha|^q > t^{q/p}) \leq t^{-q/p} E|X_\alpha|^q$$

for all α . Therefore,

$$\sup_\alpha (1 - F_\alpha(t^{1/p})) \leq \max(1, ct^{-q/p}),$$

where $c = \sup_\alpha E|X_\alpha|^q$. Because the right side above is integrable, the claim follows by Proposition 2. \square

5 Summary

This section concludes the paper with a diagram (Figure 1) which summarizes the relationships between the various stochastic boundedness properties. To illustrate how these concepts relate to compactness properties of probability measures, let us recall the definitions of Prohorov and Wasserstein metrics.

The *Prohorov metric* on the space M of probability measures on \mathbb{R}^d is defined by

$$d_P(\mu, \nu) = \inf \{ \epsilon > 0 : \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ and } \nu(B) \leq \mu(B^\epsilon) + \epsilon \text{ for all } B \in \mathcal{B} \},$$

where $B^\epsilon = \{x \in \mathbb{R}^d : |x - b| < \epsilon \text{ for some } b \in B\}$ denotes the ϵ -neighborhood of B , and \mathcal{B} denotes the Borel sets of \mathbb{R}^d . The space (M, d_P) is a complete separable

metric space, and convergence with respect to d_P corresponds to convergence in distribution (e.g. [4, Thm. 3.1.7]).

For $p \geq 1$, denote by M_p the space of probability measures on \mathbb{R}^d with a finite p -th moment. The p -Wasserstein metric on M_p is defined by

$$d_{W,p}(\mu, \nu) = \left(\inf_{\gamma \in K(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \gamma(dx, dy) \right)^{1/p},$$

where $K(\mu, \nu)$ is the set probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν . The space $(M_p, d_{W,p})$ is a complete separable metric space, and a sequence converges with respect to $d_{W,p}$ if and only if it is uniformly p -integrable and converges in distribution [1, Prop. 7.1.5]. See Rachev and Rüschendorf [15] for a detailed account of dual characterizations of $d_{W,p}$, and Gibbs and Su [5] for a nice survey of different probability metrics and their relationships.

Theorem 3. *For any $0 < p < 1 < q < \infty$ and for any family of random variables $\{X_\alpha\}$ in \mathbb{R}^d with probability distributions $\mu_\alpha = P \circ X_\alpha^{-1}$, the implications in Figure 1 are valid. In general, no other implications hold.*

Proof. Let us start by verifying the equivalences on the left of Figure 1, starting from top. The first and fourth are due to Proposition 2. The second, third, and fifth are due to Theorem 1. The sixth is due to Proposition 1.

Let us next verify the equivalences on the right, again starting from the top. The first is due to Theorem 2. By applying Ambrosio, Gigli, and Savaré [1, Prop. 7.1.5], we know that $\{\mu_\alpha\}$ is relatively compact in the q -Wasserstein metric if and only if $\{X_\alpha\}$ is uniformly q -integrable and tight. Because closed bounded sets are compact in \mathbb{R}^d , it follows that uniform q -integrability implies tightness. Therefore, the second equivalence on the right holds. The third equivalence follows by observing that the L^q -norm of X_α is equal to the q -Wasserstein distance between μ_α and the Dirac measure at zero. The fourth and fifth equivalences are verified by similar reasoning. For the sixth, it suffices to recall that tightness and relative compactness in the Prohorov metric are equivalent by Prohorov's classic theorem (e.g. [8, Thm 16.3]).

The downward implications are more or less immediate. Starting from top, the first, fourth, and seventh are trivial, because strong ordering implies increasing convex ordering. The second, fifth, and eighth are trivial as well, because increasing convex ordering implies the ordering of the means. The third, sixth, and ninth are due to Proposition 3.

To complete the picture, we will now construct families of random variables which show that none of the one-way implications in Figure 1 can be reversed in general. Let U be a uniformly distributed random variable in $(0, 1)$, and let ϕ_n and ψ_n , $n \geq 2$, be random variables such that

$$\phi_n = \begin{cases} n & \text{with probability } n^{-1}, \\ 0 & \text{else,} \end{cases} \quad \psi_n = \begin{cases} n & \text{with probability } (n \log n)^{-1}, \\ 0 & \text{else.} \end{cases}$$

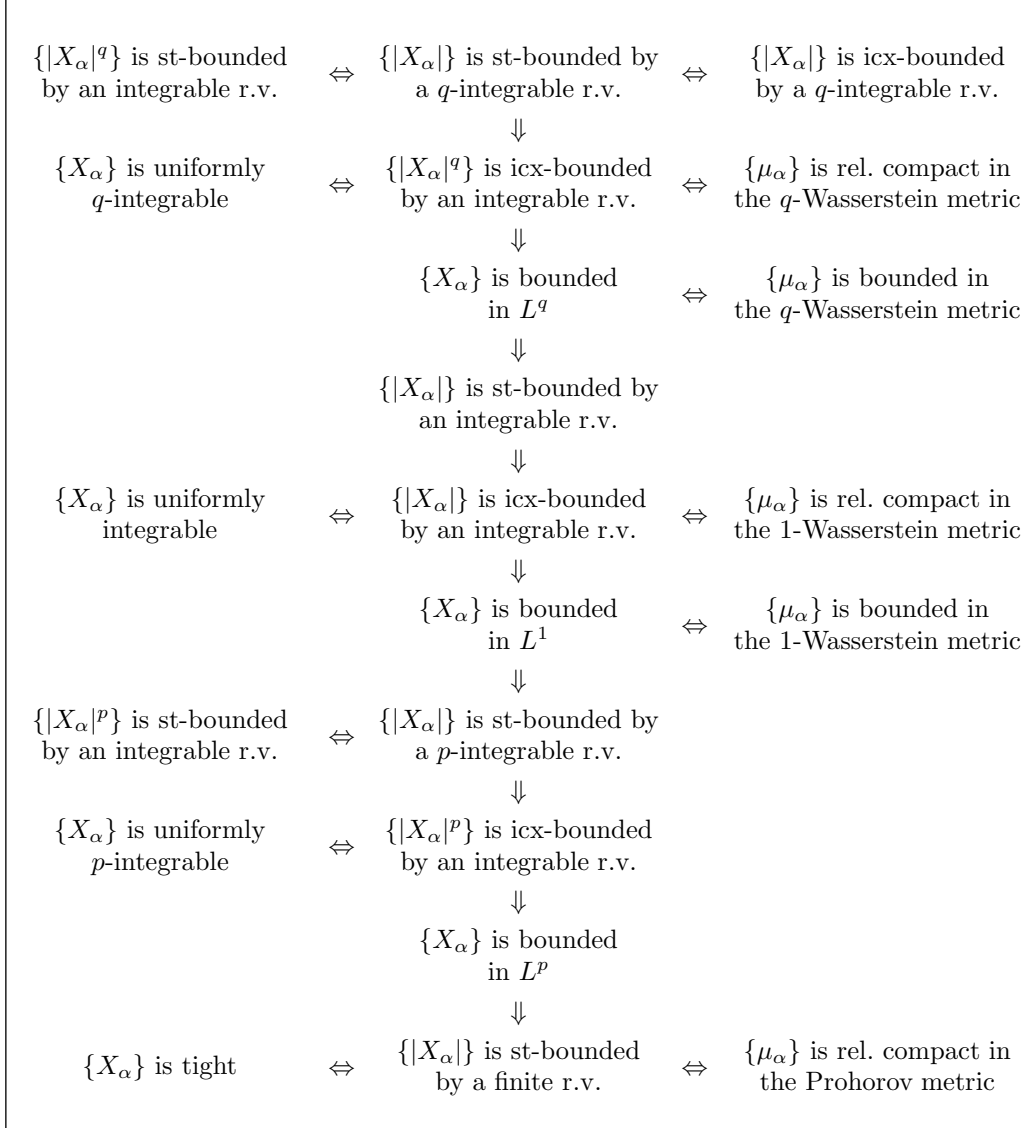


Figure 1: Stochastic boundedness relationships.

Then for any $0 < p < 1 < q$:

- $\{e^{1/U}\}$ is st-bounded by a finite r.v. but not bounded in L^p .
- $\{\phi_n^{1/p}\}$ is bounded in L^p but not uniformly p -integrable.
- $\{\psi_n^{1/p}\}$ is uniformly p -integrable but not st-bounded by a r.v. in L^p .
- $\{U^{-1}\}$ is st-bounded by a r.v. in L^p but not bounded in L^1 .

- $\{\phi_n\}$ is bounded in L^1 but not uniformly integrable.
- $\{\psi_n\}$ is uniformly integrable but not st-bounded by an integrable r.v.
- $\{U^{-1/q}\}$ is st-bounded by an integrable r.v. but not bounded in L^q .
- $\{\phi_n^{1/q}\}$ is bounded in L^q but not uniformly q -integrable.
- $\{\psi_n^{1/q}\}$ is uniformly q -integrable but not st-bounded by a r.v. in L^q .

The straightforward computations required for verifying the above claims are left to the reader. \square

A Distribution functions

If X is a finite positive random variable, then its distribution function $F(t) = P(X \leq t)$ is increasing, right-continuous, and satisfies $F(0) \geq 0$ and $\lim_{t \rightarrow \infty} F(t) = 1$. Recall that any function F with these properties can be realized as the distribution function of the random variable $X = F^{-1}(U)$, where $F^{-1}(u) = \inf\{t \in \mathbb{R}_+ : F(t) \geq u\}$ and U is a uniformly distributed random variable in $(0, 1)$.

Lemma 2. *For any family $\{F_\alpha\}$ of distribution functions on \mathbb{R}_+ , the function $F(t) = \inf_\alpha F_\alpha(t)$ is a distribution function on \mathbb{R}_+ if and only if*

$$\lim_{t \rightarrow \infty} \inf_\alpha F_\alpha(t) = 1. \quad (5)$$

Proof. The necessity of (5) is obvious. To prove sufficiency, assume that a family of distribution functions $\{F_\alpha\}$ on \mathbb{R}_+ satisfies (5). Then it immediately follows that F is increasing, $F(0) \geq 0$, and $\lim_{t \rightarrow \infty} F(t) = 1$. Therefore, we only need to verify that F is right-continuous. Fix arbitrary $t \geq 0$ and $\epsilon > 0$, and choose an index α such that $F_\alpha(t) < F(t) + \epsilon/2$. By the right-continuity of F_α there exists a $\delta > 0$ such that $F_\alpha(t+h) - F_\alpha(t) < \epsilon/2$ for all $h \in (0, \delta)$. As a consequence,

$$F(t+h) \leq F_\alpha(t+h) < F_\alpha(t) + \epsilon/2 < F(t) + \epsilon$$

for all $h \in (0, \delta)$. Because F is increasing, this implies that F is right-continuous. \square

B Integrated survival functions

The integrated survival function of a positive integrable random variable X with distribution function F is defined by

$$H(t) = E(X - t)_+ = \int_t^\infty (1 - F(u)) du.$$

The following result characterizes the family of integrated survival functions generated by positive integrable random variables. The proof is similar to an analogous characterization for random variables on the full real line (Müller and Stoyan [14, Thm. 1.5.10]). Because our formulation of the result is slightly different, we include the proof here for the reader's convenience.

Lemma 3. *A function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the integrated survival function of a positive integrable random variable if and only if*

- (i) H is convex,
- (ii) $\lim_{t \rightarrow \infty} H(t) = 0$,
- (iii) $H(t) + t \geq H(0)$ for all $t \geq 0$.

Proof. Assume first that H is the integrated survival function of a positive integrable random variable X . Then (i) follows immediately because $t \mapsto (x - t)_+$ is convex for all x , and (ii) follows by dominated convergence. To see the validity of (iii), it suffices to observe that

$$H(t) + t = E \max(X, t) \geq EX = H(0).$$

Assume next that a function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies (i)–(iii). The convexity of H together with (ii) implies that H is decreasing on \mathbb{R}_+ . By applying (iii) we see that $-t \leq H(t) - H(0) \leq 0$ for all t , which implies that H is continuous at zero. As a consequence, the right derivative $H'_+(t)$ of H exists for all $t \geq 0$, and the function $H'_+ : \mathbb{R}_+ \rightarrow \mathbb{R}$ is right-continuous and increasing (e.g. [2, Prop. 4.1.1]). Define $F(t) = 1 + H'_+(t)$. Then $F(0) \geq 0$ due to (iii), and $\lim_{t \rightarrow \infty} F(t) = 1$, because $H(t)$ decreases to zero as $t \rightarrow \infty$. Because F is right-continuous and increasing, we conclude that F is the distribution function of the random variable $X = F^{-1}(U)$, where U is uniformly distributed in $(0, 1)$. To see that H is the integrated survival function of X , we note by changing the order of integration that for all t ,

$$H(t) = - \int_t^\infty H'_+(u) du = \int_t^\infty (1 - F(u)) du = E(X - t)_+.$$

Especially, $EX = H(0)$ is finite, which shows that X is integrable. \square

Lemma 4. *For any integrable positive random variable X and for any $p > 1$,*

$$EX^p = p \int_0^\infty H(t^{1/(p-1)}) dt,$$

where $H(t) = E(X - t)_+$.

Proof. Denote $G(u) = P(X > u)$. Because $P(X^p > u) = G(u^{1/p})$, a change of variables shows that

$$EX^p = \int_0^\infty G(u^{1/p}) du = p \int_0^\infty G(u) u^\epsilon du,$$

where $\epsilon = p - 1$. Further, by writing $u^\epsilon = \int_0^\infty 1(t^{1/\epsilon} < u) dt$ and changing the order of integration, we find that

$$p \int_0^\infty G(u) u^\epsilon du = p \int_0^\infty \left(\int_{t^{1/\epsilon}}^\infty G(u) du \right) dt = p \int_0^\infty H(t^{1/\epsilon}) dt.$$

□

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